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Critical withdrawal from a two-layer fluid through a line sink

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Abstract. The problem of withdrawing water through a line sink from a region containing two homogenous layers of different density is considered. Assuming steady, irrotational flow of an ideal fluid, a nonlinear integral equation is derived and solved numerically. Confirmation of earlier research is given, and some new results obtained in which the interface between the two layers rises up and then enters the sink vertically from above, even when the sink is located above the undisturbed level of the interface. A diagram is presented which summarises the work on this problem to this time.

1. Introduction

There are many examples of water bodies which are stratified in density. Water stored in reservoirs is often stratified into layers of different temperature or salinity, and hence density, by the action of the weather on the surface and the inflowing rivers. This stratification has a rather irregular structure, although it typically consists of a reasonably well mixed surface layer several metres thick, above a sharp density interface, which in turn lies above a weakly stratified lower layer reaching down to the bottom [12].

Water in power station cooling ponds, and water in solar ponds used for power generation, are two examples in which there is a more clearly defined layering of the water body. Cooling ponds usually have a layer consisting of warm, recently used water, above a cooler, more dense layer. In solar ponds, power is generated by building a two-layer system of cold, fresh water above warm, saline water. The lower layer then traps the solar radiation and heats up, storing the energy. A discussion of the physics of these water bodies can be found in [12].

In all of these examples, it is of interest to know the behaviour of the flow pattern induced when water is withdrawn. In a reservoir this knowledge allows prediction (and sometimes selection) of water quality, while in a cooling pond it allows the cooling process to be carried out with maximum efficiency by allowing some control of the temperature of the withdrawn water. Energy is extracted from a solar pond by withdrawing water from the hot, saline layer, and withdrawals and inflows are used to maintain the most efficient density gradient for the collection and storage of energy in the pond.

The work presented here examines the problem of withdrawal through a line sink from two distinct, homogeneous layers separated by a very sharp interface. In reality, the water bodies described may not have such a convenient structure, and the actual situation may be some combination of the two-layer case and the case of withdrawal from a fluid in which the density stratification is linear. The latter problem will not be discussed here, but a recent review of this subject can be found in Imberger and Patterson [15].

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Apart from improving fundamental understanding of the physics of withdrawal, the results can be implemented in numerical models for simulating density structure in lakes, reservoirs, cooling ponds and solar ponds, of the type discussed in [2], [8], [13], [14] and [19]. These models are extremely useful for developing reservoir management strategies and for forecasting the behaviour of water in proposed works.

When water is withdrawn from a fluid consisting of two or more homogeneous layers of different density, separated by an interface, the withdrawn water will come from the layer adjacent to the point of removal until some threshold in flow rate is reached, after which the other two layers will begin to flow out through the sink. This threshold can be described most easily in terms of a dimensionless parameter, the Froude number,

$$F = \left(\frac{q^2}{g' h_b^3}\right)^{1/2},$$

where q is the discharge through the sink per unit width, h_b is the depth of the lower layer of fluid, and g' is $(\Delta\rho/\rho)g$ where ρ is the density of the lower layer, $\Delta\rho$ is the difference in density between the two layers, and g is the acceleration due to gravity.

If the interface is assumed to be of infinitesimal thickness, the flow to be steady and irrotational, and the fluid to be inviscid and incompressible, it can be shown (Tuck and Vanden-Broeck [20]), that only two types of solution are possible beneath the threshold Froude number, i.e. when only a single layer is flowing out through the sink. We shall suppose the sink to be in the lower layer, and hence this is the only layer from which water is being withdrawn at Froude numbers beneath the threshold value.

The first solution type involves a stagnation point on the interface directly above the sink. Peregrine [18], Vanden-Broeck, Schwartz and Tuck [22], and Tuck and Vanden-Broeck [20], all computed solutions of this type for the case of a line sink, but had limited success as the Froude number was increased. It was suggested by their work that there exists a critical value of F above which such solutions do not exist. Recently, Hocking and Forbes [9], computed solutions with a stagnation point for the case of a 'layer' of infinite depth, for values of Froude number up to 1.4. In this infinite depth case, the Froude number must be redefined as $F_s = (q/(g'h_s^3))^{1/2}$, where h_s is the depth of the sink beneath the level of the interface. No solutions of this type were found for values of the Froude number greater than this.

Forbes and Hocking [5] have obtained solutions in the case of axisymmetric flow into a point sink, for values of F_{3s} up to approximately 6.4. In three dimensions the Froude number is defined as $F_{3s} = (Q^2/g'h_s^5))^{1/2}$ where Q is the total flux into the sink. At $F_{3s} = 6.4$, they found a secondary stagnation point formed on the interface a small distance away from the primary point above the sink. As in the case of a line sink, no solutions were obtained for values of F_{3s} larger than this.

The second solution type has long been thought to characterise the threshold value of the Froude number, above which water in the layer above the interface begins to be drawn out through the sink. In this flow, a downward cusp forms in the interface directly above the sink, as the interface is drawn down to enter the sink vertically (see Fig. 1). It is this type of solution, induced by the flow of a line sink, which is the major interest of the work described here.

In the situation in which the lower layer (containing the sink) extends downward to infinity, it has been shown on numerous occasions (Craya [3], Tuck and Vanden-Broeck [20],

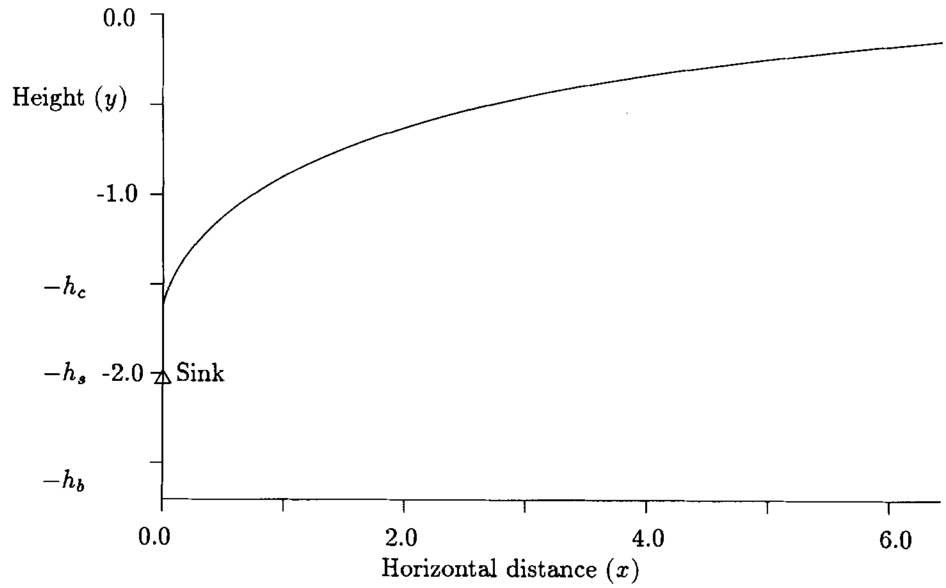

Fig. 1. Definition sketch for the problem under consideration. The interface shape shown was computed for the case $h_s/h_b = 0.75$ and $F = 1$.

Hocking [6]), for various different geometries, that solutions with this cusp exist only at a single value of the Froude number. No solutions of this type have been found at values of F_s slightly smaller, and none for values of F_s slightly larger. This reinforced the belief that this solution characterised the threshold flow, and that at slightly larger values of F_s the upper layer would also be drawn into the sink.

Yih [24] speculated, however, that when the lower layer is of finite depth solutions of this shape might exist for flows in the absence of gravity, i.e. at infinite values of Froude number, and hence that solutions at large but finite values of F would also exist. Solutions with infinite Froude number were subsequently found by Collings [4], Vanden-Broeck and Keller [21], Hocking [7] and King and Bloor [7].

Vanden-Broeck and Keller [21] went further, and used a series solution method to obtain solutions with a cusp for Froude numbers ranging from infinity down to a value greater than or equal to unity, in a fluid of finite depth, with the sink situated at various depths in the lower fluid (see Fig. 1). They suggested that further solutions with waves on the interface might exist for values of Froude number less than one, although they did not compute them. An extra branch of waveless solutions was found for Froude numbers less than one. For each of a range of small sink to bottom depth ratios, a single cusped interface solution was obtained which proved to be unique for the ratio (see Fig. 4). This branch of solution contains the solution of Tuck and Vanden-Broeck [20] for a fluid of infinite depth.

In the case of infinite depth, the results of Hocking and Forbes [9], who found stagnation point solutions for F_s up to 1.4, and Tuck and Vanden-Broeck [20], who found a single cusped solution at $F_s = 3.54$, reveal the presence of a gap in the parameter range. It is possible that some kind of unsteady transition flow exists in the range $1.4 < F_s < 3.54$. In a real situation, the question arises as to whether the drawdown could occur in this 'transition zone', or even in the region of the breakdown of the stagnation point flows. Experimental work [10, 16] indicates that the drawdown Froude number is lower than that predicted by numerical solution of the idealised problem.

In this paper, an integral equation is derived and solved which verifies all of the results of Vanden-Broeck and Keller [21], and extends the range of solution to include the slightly surprising case in which the sink is situated above the interface level at large distances from the sink, but is drawing fluid only from the lower layer, shown by Hocking [7] and King and Bloor [17], to have solutions in the absence of gravity. This flow could arise if water was being withdrawn very quickly from a large tank, and the level of the interface away from the sink fell slowly to beneath the level of the sink.

The problem is formulated in such a way that the results could apply equally to withdrawal from a fluid having a free surface (g' is replaced by g in the Bernoulli equation), or to a source flow rather than a sink. It is not possible to distinguish between the latter two flow types, because of the quadratic dependence of the velocity in the boundary condition, i.e. the condition is independent of the flow direction.

A diagram is presented which summarises the known solutions with a cusped interface to the problem of withdrawal from a two layer fluid through a line sink. The limits on the stagnation point flows at low Froude number are now shown, since they are only known for the case of infinite depth. It is hoped that with further research the gaps in the diagram can be filled.

2. Problem formulation and solution

The steady, irrotational motion of an inviscid, incompressible fluid in the presence of gravity in two dimensions is to be examined. The fluid is of finite depth and has a density interface above a line sink.

Let $z = x + iy$ be the physical plane, with the origin directly above the sink and at the level of the interface far away from the sink (see Fig. 1). The mathematical problem is to find a complex potential $w = \phi(x, y) + i\psi(x, y)$, which satisfies Laplace's equation ($\nabla^2 w = 0$) within the flow domain (the lower layer), conditions of no flow across the solid boundaries and the interface, and the condition of constant pressure on the interface, provided by Bernoulli's equation

$$p = \rho g' y + \frac{1}{2}\rho\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right) = \frac{1}{2}\rho U^2 \tag{2.1}$$

on $y = \eta(x)$ where $\eta(x)$ is the equation of the interface shape, and U is the velocity of the fluid far from the sink. If we nondimensionalise with respect to the length $(m^2/18\pi^2 g')^{1/3}$ and the velocity $(2mg'/3\pi)^{1/3}$, where m is the sink strength, then this equation becomes

$$y + \left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right) = \left(\frac{\pi}{h_b}\right)^2, \tag{2.2}$$

where h_b is the depth of the fluid at large distances from the sink, the nondimensional flux into the sink is π, and hence the velocity at large distances from the sink is π/h_b. Since the Bernoulli equation has a quadratic dependence upon the velocity, the equations are equally valid for a source flow, and the boundary conditions are formulated in this way. The flow is symmetric about $x = 0$, and consequently only the region $x \geq 0$ is considered.

To derive an integral equation for this problem we follow a similar procedure to that used in Hocking [7]. The transformation

$$e^w = \zeta \tag{2.3}$$

maps the infinite strip between $\psi = 0$ and $\psi = -\pi$ in the w-plane to the lower half of the ζ-plane. Without loss of generality we may choose to let $w = 0$ correspond to the cusp point, so that the free surface $y = 0$, $\phi > 0$, lies along the real ζ-axis where $\zeta \geq 1$. The sink lies at the origin in the ζ-plane, and the negative real axis corresponds to $\psi = -\pi$ (see Fig. 2).

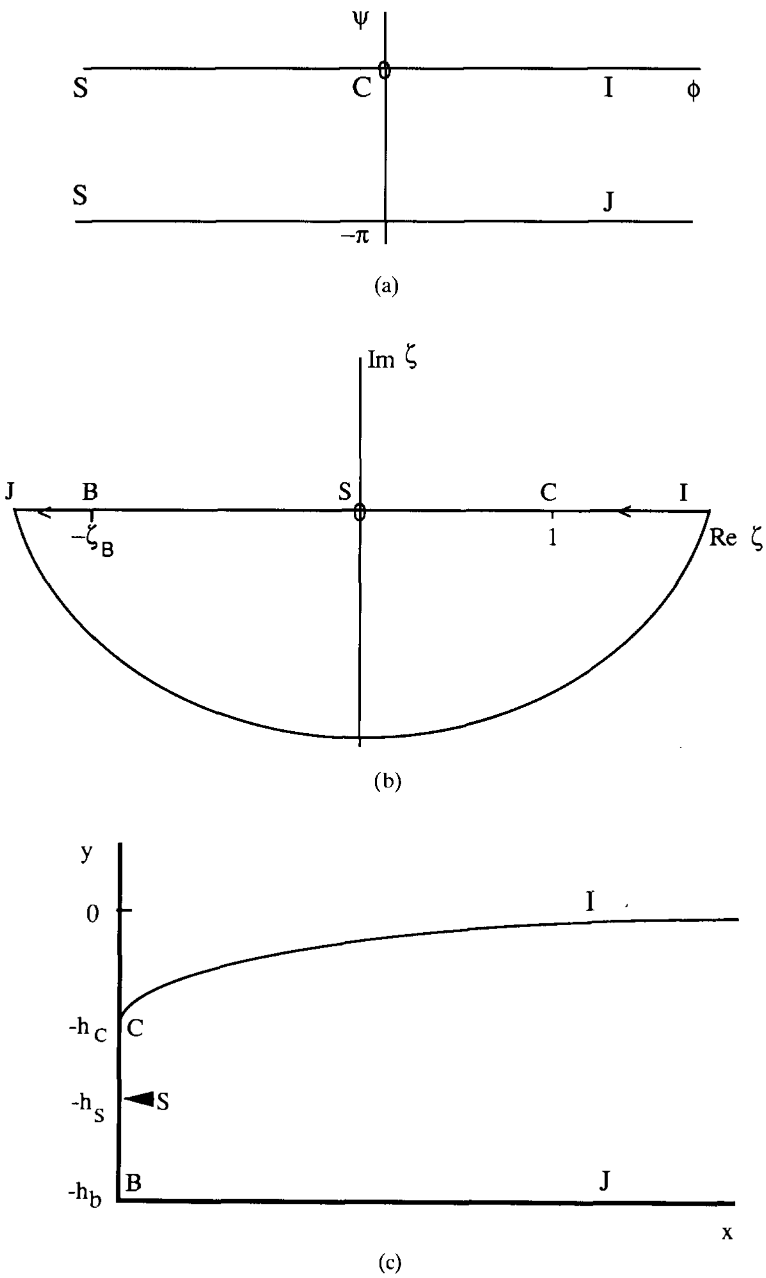

Fig. 2. Mapped planes used in the problem formulation; (a) the complex velocity potential w-plane, (b) the lower half ζ-plane, and (c) the physical z-plane.

We seek w by solving for $\Omega(\zeta)=\delta(\zeta)+i\tau(\zeta)$, defined in relation to the complex conjugate of the velocity field by

$$w'(z(\zeta))=\left(\frac{\pi}{h_b}\right)e^{-i\Omega(\zeta)}\,. \tag{2.4}$$

The magnitude of the velocity at any point is then given by $|w'(z)|=(\pi/h_b)e^{\tau(\zeta)}$, and the angle any streamline makes with the horizontal is $\delta(\zeta)$. Thus, for cusp-like solutions, we require that $\delta=\pi/2$ at $\zeta=1$ and $\delta\to 0$ as $\zeta\to\infty$.

The interface corresponds to the positive real ζ-axis for $\zeta>1$, and $\tau\to 0$ as $\zeta\to\infty$ to conserve mass. On the remainder of the real ζ-axis, which corresponds to the solid boundaries of the flow domain, the streamlines must be parallel to the walls, so that the condition that there be no flow normal to the solid boundaries is satisfied if we choose $\delta(\zeta)$ to be the angle of the wall to the horizontal, i.e.

$$\delta(\zeta)=\begin{cases}0 & \text{if } -\infty<\zeta<\zeta_B\,;\\ -\pi/2 & \text{if } \zeta_B<\zeta<0\,;\\ \pi/2 & \text{if } 0<\zeta\leqslant 1\,.\end{cases}$$

The only singularities of the function $\Omega(\zeta)$ in the ζ-plane are those at the origin and at $\zeta=\zeta_B$, corresponding to the sink and the stagnation point on the bottom beneath the sink respectively. Both of these singularities can be shown to be weaker than a simple pole, so that Cauchy's Theorem can be applied to $\Omega(\zeta)$ on a path consisting of the real ζ-axis a semi-circle at $|\zeta|=\infty$ in the lower half plane, and a circle of vanishing radius about the point ζ. Hence, for $\Im m\{\zeta\}<0$ we have

$$\Omega(\zeta)=-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Omega(\zeta_0)}{\zeta_0-\zeta}\,d\zeta_0\,, \tag{2.5}$$

since $\Omega\to 0$ as $|\zeta|\to\infty$. If we let $\Im m\{\zeta\}\to 0^-$, we obtain

$$\tau(\zeta)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\delta(\zeta_0)}{\zeta_0-\zeta}\,d\zeta_0$$

and

$$\delta(\zeta)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\tau(\zeta_0)}{\zeta_0-\zeta}\,d\zeta_0\,, \tag{2.6}$$

where the integrals are of Cauchy Principal Value form.

Substituting the known values of $\delta(\zeta)$ into the equation for $\tau(\zeta)$ gives

$$\tau(\zeta)=\frac{1}{2}\ln\left(\frac{(1-\zeta)(\zeta_B-\zeta)}{\zeta^2}\right)+\frac{1}{\pi}\int_1^{\infty}\frac{\delta(\zeta_0)}{\zeta_0-\zeta}\,d\zeta_0\,. \tag{2.7}$$

The singularity at $\zeta=1$ in (2.7) can be removed by noting that

$$-\frac{1}{\pi}\int_1^{\infty}\frac{\arcsin\zeta_0^{-1/2}}{\zeta_0-\zeta}\,d\zeta_0=\frac{1}{2}\ln\left(\frac{1-\zeta}{\zeta}\right), \tag{2.8}$$

and writing $\delta(\zeta)=\arcsin\zeta^{-1/2}+\delta_b(\zeta)$, so that (2.7) becomes

$$\tau(\zeta) = \frac{1}{2} \ln\left(\frac{\zeta_B - \zeta}{\zeta}\right) + \frac{1}{\pi} \int_1^{\infty} \frac{\delta_b(\zeta_0)}{\zeta_0 - \zeta} \, d\zeta_0 \, . \tag{2.9}$$

To this we must add the equation for constant pressure on the interface, which can be obtained by combining equations (2.2), (2.3) and (2.4) to give

$$\frac{h_b}{\pi} \int_{\infty}^{\zeta} \frac{e^{-\tau(\zeta_0)} \sin \delta(\zeta_0)}{\zeta_0} \, d\zeta_0 + \left(\frac{\pi}{h_b}\right)^2 e^{2\tau(\zeta)} = \left(\frac{\pi}{h_b}\right)^2 , \tag{2.10}$$

on $1 \leqslant \zeta < \infty$. This equation can be differentiated, rearranged and integrated to give the more convenient form

$$\tau(\zeta) = \frac{1}{3} \ln\left(1 + \frac{3h_b^3}{2\pi^2} \int_{\zeta}^{\infty} \frac{\sin \delta(\zeta_0)}{\zeta_0} \, d\zeta_0\right) , \tag{2.11}$$

on $1 \leqslant \zeta < \infty$. Combining (2.9) and (2.11) on the interface gives a nonlinear integral equation for $\delta(\zeta)$ on $1 \leqslant \zeta < \infty$. The value of δ is known elsewhere on the boundary from the boundary conditions, and hence we can obtain τ from (2.6). Using δ and τ, it is possible to integrate (2.4) to obtain the location of points on the interface. These may be written as

$$x(\zeta) = x(\zeta^*) + \frac{h_b}{\pi} \int_{\zeta^*}^{\zeta} \frac{e^{-\tau(\zeta_0)} \cos \delta(\zeta_0)}{\zeta_0} \, d\zeta_0 \, ,$$

and

$$y(\zeta) = y(\zeta^*) + \frac{h_b}{\pi} \int_{\zeta^*}^{\zeta} \frac{e^{-\tau(\zeta_0)} \sin \delta(\zeta_0)}{\zeta_0} \, d\zeta_0 \, . \tag{2.12}$$

Since $y \to 0$ as $\zeta \to \infty$, the cusp depth is

$$h_C = \frac{h_b}{\pi} \int_1^{\infty} \frac{e^{-\tau(\zeta_0)} \sin \delta(\zeta_0)}{\zeta_0} \, d\zeta_0 \, , \tag{2.13}$$

the sink depth is

$$h_s = h_C + \frac{h_b}{\pi} \int_0^1 \frac{e^{-\tau(\zeta_0)}}{\zeta_0} \, d\zeta_0 \, , \tag{2.14}$$

and the base depth is

$$h_b = h_s + \frac{h_b}{\pi} \int_0^{\zeta_B} \frac{e^{-\tau(\zeta_0)}}{\zeta_0} \, d\zeta_0 \, . \tag{2.15}$$

3. Numerical method

The nonlinear integral equation described by equations (2.9) and (2.11) has no closed-form solution, but can be solved by computing the integrals numerically at a set of discrete points and solving for $\delta(\zeta)$ using Newton's iteration method.

Since $e^w = \zeta$, $e^\phi = \zeta$ on $\psi = 0$, i.e. $\phi = \ln \zeta$. It is convenient to integrate with respect to ϕ rather than ζ, since this choice crowds the points near the region of greatest change in the ζ-plane, i.e. near the cusp point. The integral to infinity was truncated at some large number $\zeta_t = e^{\phi_t}$, and points were chosen at $N+1$ equally spaced values of ϕ, so that $0 = \phi_0 < \phi_1 < \phi_2 < \cdots < \phi_N = \phi_t$, with point spacing $\Delta\phi = (\phi_N - \phi_0)/N$. The integral equation consisting of (2.9) and (2.11) was evaluated at the mesh points ϕ_j, for $j = 1, 2, \ldots, N$. If the last term of (2.7) is written as

$$\frac{1}{\pi}\int_1^{\zeta_t} \frac{\delta(\zeta_0)}{\zeta_0 - \zeta}\, d\zeta_0 = \frac{1}{\pi}\int_1^{\zeta_t} \frac{\delta(\zeta_0) - \delta(\zeta)}{\zeta_0 - \zeta}\, d\zeta_0 + \frac{\delta(\zeta)}{\pi} \ln\left(\frac{\zeta_t - \zeta}{\zeta - 1}\right), \tag{3.1}$$

then it is no longer singular at $\zeta = \zeta_0$ and can be integrated without difficulty. Cubic splines were used for all of the calculations in this paper.

In order to minimise the error caused by the truncation of the integral at ϕ_t, δ was assumed to decay like $e^{-\frac{1}{2}\phi}$, i.e. $\zeta^{-1/2}$, as in the infinite Froude number case (see [7] for example), and an exact calculation was made to account for the reduction by fitting this function to the last value of δ, that is $\delta(\phi_N)$. This also reduces the error in using (3.1) as ζ approaches ζ_t.

The boundary condition at $\zeta = 1$ gives $\delta_0 = \pi/2$, leaving N equations for the N unknown values of δ_j, $j = 1, 2, \ldots, N$. The value of ζ_B is unknown, however, for a channel of given depth and to close the system we need another equation. This is obtained by fixing the sink depth, i.e. using (2.14). $\tau(\phi_i)$ can be calculated from the previous guess for $\delta(\phi_j)$, $j = 1, 2, \ldots, N$.

This closed system of $N+1$ nonlinear algebraic equations can be solved using Newton's method. Using a first guess for δ given by the infinite Froude number solution (Hocking, [7]), convergence to an error of less than 10^{-9} at the grid points was usually achieved within five iterations. The numerical values of sink and cusp depth converged to three figure accuracy with $N = 160$.

In each run of the numerical scheme, the value of F was fixed and the value of h_s/h_b gradually decreased, using the previous solution for δ_j as an initial guess, until the Newton's method failed to converge.

To compute the waveless solutions with $F < 1$ obtained by Vanden-Broeck and Keller [21], it was necessary to make some minor alterations to the numerical scheme. Since for each value of F less than one a solution only exists for a single value of h_s/h_b, F was fixed and h_b was allowed to vary, i.e. become an output of the computations. This in turn necessitated reducing the number of variables by one (since the number of equations has been reduced by one), and this was achieved by letting $\delta_N = a\delta_{N-1}$. Numerical experiments showed the results to be independent of the choice of a, in the range $0 < a < 1$, and a value of $a = \frac{1}{2}$ was used.

4. Results

The method described was used successfully to repeat all of the work of Vanden-Broeck and Keller [21] to graphical accuracy. The branch of solution in which the interface contained a hump above the sink as the sink depth approached zero was successfully continued into the region where the sink rises above the level of the interface in the far field. An example of an interface shape for a flow of this type is shown in Fig. 3. There did not appear to be any limit

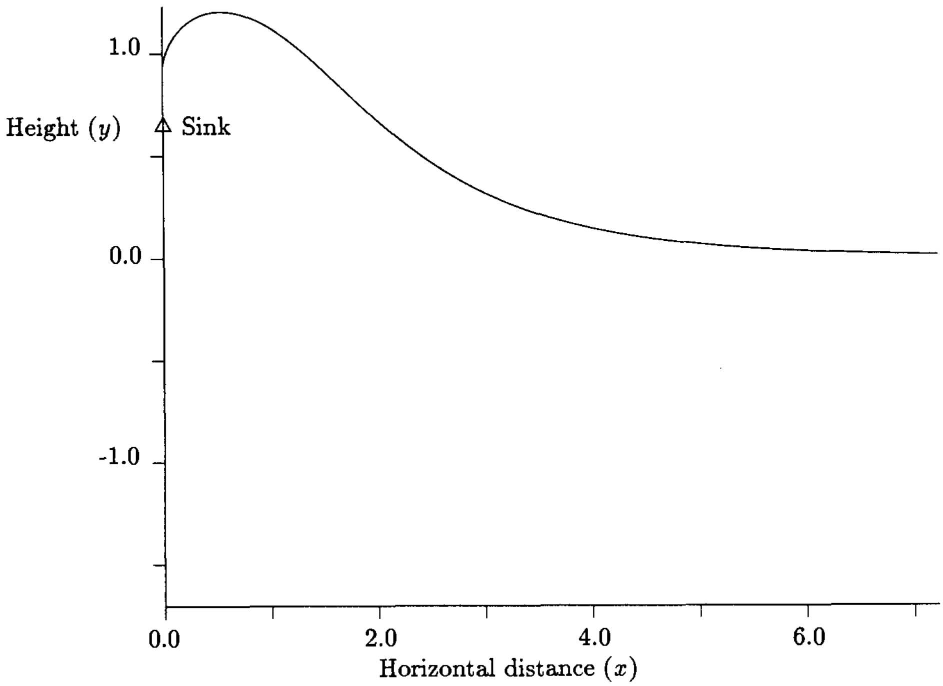

Fig. 3. Diagram showing the interface shape for a situation in which the sink is above the level of the interface away from the sink ($F = 2$ and $h_s/h_b = -0.38$).

to such solutions as the Froude number was increased, and in fact we know from Hocking [7] and King and Bloor [17], that infinite Froude number solutions exist for all of these geometries. For each value of h_s/h_b the lowest value of Froude number at which cusped solutions were obtained was noted, and the results are shown in Fig. 4.

Attempts were made to compute solutions with a cusp for values of Froude number less than unity, apart from the waveless branch found by Vanden-Broeck and Keller [21], without success. This failure may be due to the presence of waves on the interface at large distances from the sink, as suggested by Vanden-Broeck and Keller [21], or it may be that no solutions with a cusp exist for Froude numbers less than one.

Figure 4 summarises those solutions which are known of at the time of writing this paper. Solutions with a stagnation point for small values of Froude number are not included since it is not known what the limiting value on their existence is, except in the infinite depth problem. Peregrine [18] suggested that the breakdown of such solutions might occur when a limiting configuration with a 120 degree angle at the stagnation point formed. No sign of this type of flow was found by Hocking and Forbes [9] for the infinite depth case, although preliminary results by Hocking [11] indicate their existence in the case of a layer of finite depth.

Whilst it seems unlikely that the parameter range in which there are flows with a cusped interface would overlap with the range in which there are flows with a stagnation point, it is possible if one flow type were to correspond to a sink flow, and the other to a source flow. An added complication to this is the fact that there appear to be two branches of solution in which the flow has a cusp shape, and it is possible that one of these might occur for a sink and the other for a source. The solutions with $h_s/h_b < 0$ look more realistic for a source flow, yet mathematically they are possible for a sink flow as well. There is no way of differentiat-

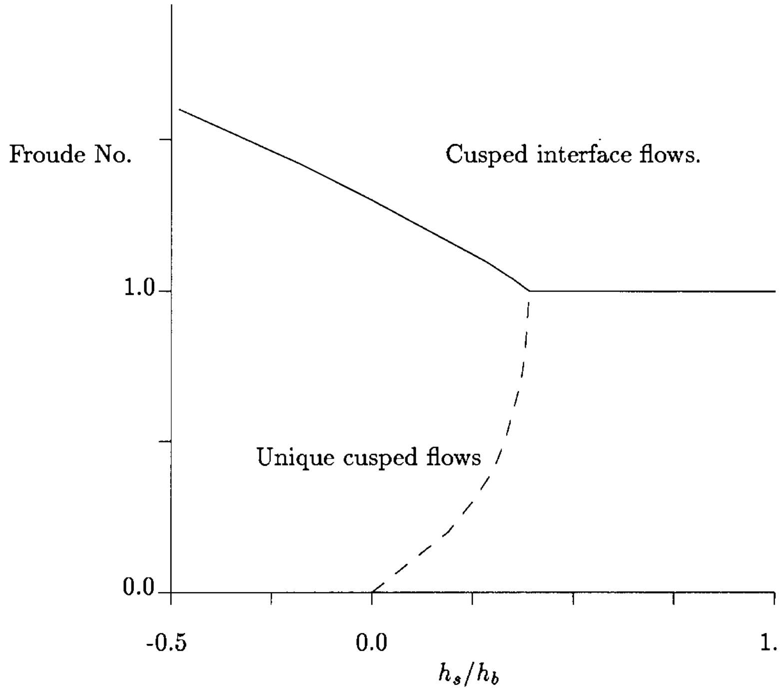

Fig. 4. Regions in the parameter space in which solutions with a cusp have been computed. Stagnation point solutions exist in the region $F < 1$, but the bounds on them, except in the case $h_s/h_b = 0$, are unknown, and consequently they are now shown.

ing between the source and sink using the formulation presented here, and consequently this issue must remain unresolved for the moment. Work is continuing to complete Fig. 4 for all flow types.

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References

1. Abramowitz, M. and Stegun I.A. eds, *Handbook of Mathematical Functions*, Dover, New York (1970).
2. "CE-QUAL-R1: A numerical one-dimensional model of reservoir water quality: User's manual." *Instruction Rept. E-82-1*, Envir. Lab., U.S. Army Engr. Wtrwy. Expt. Sta., CE, Vicksburg, Miss. (1982).
3. Craya, A., Theoretical research on the flow of nonhomogeneous fluids, *La Houille Blanche* 4 (1949) 44–55.
4. Collings, I.L., Two infinite Froude number cusped free surface flows due to a submerged line source or sink, *J. Aust. Math Soc. Ser. B* 28 (1986) 260–270.
5. Forbes, L.K. and Hocking, G.C., Flow caused by a point sink in a fluid having a free surface, *J. Aust. Math Soc. Ser. B* 32 (1990) 233–252.
6. Hocking, G.C., Cusp-like free-surface flows due to a submerged source or sink in the presence of a flat or sloping bottom, *J. Aust. Math Soc. Ser. B* 26 (1985) 470–486.

7. Hocking, G.C., Infinite Froude number solutions to the problem of a submerged source or sink, *J. Aust. Math. Soc. Ser. B* 29 (1988) 401–409.
8. Hocking, G.C., Sherman, B.S., and Patterson, J.C. (1988), Algorithm for selective withdrawal from stratified reservoir, *J. Hydr. Engng., A.S.C.E.* 114 (1988) 707–719.
9. Hocking, G.C. and Forbes, L.K., A note on the flow induced by a line sink beneath a free surface. *J. Aust. Math Soc. Ser. B* (1991) (in press).
10. Hocking, G.C., Withdrawal from a two layer fluid through a line sink, submitted to *J. Hydr. Engng., A.S.C.E.* (1989).
11. Hocking, G.C., Flow induced by a line sink in a fluid of finite depth. University of Western Australia, Dept. of Math. Res. Rept. 13, April, 1990.
12. Imberger, J. and Hamblin, P.F., Dynamics of lakes, reservoirs and cooling ponds, *Ann. Rev. Fluid Mech.* 14 (1982) 153–187.
13. Imberger, J., Reservoir Dynamics Modelling, in O'Loughlin, E.M. and Cullen, P. (eds), *Prediction in Water Quality*, Australian Acad. of Sci., Canberra, Australia (1982) 223–248.
14. Imberger, J., Patterson, J., Hebbert, R., and Loh, I., Dynamics of reservoir of medium size, *J. Hydr. Div.*, ASCE, 104 (1978) 725–743.
15. Imberger, J. and Patterson, J.C., Physical limnology, *Adv. Appl. Mech.* 27 (1989) 303–475.
16. Jirka, G.H. and Katavola D.S., Supercritical withdrawal from two-layered fluid systems, Part 2 – Three dimensional flow into a round intake, *J. Hyd. Res.* 17 (1979) 53–62.
17. King, A.C. and Bloor, M.I.G., A note on the free surface induced by a submerged source at infinite Froude number, *J. Aust. Math Soc. Ser. B* 30 (1988) 147–156.
18. Peregrine, D.H., A line source beneath a free surface, Mathematics Research Center, Univ. Wisconsin Rept. 1248 (1972).
19. Ryan, P.J. and Harleman D.R.F., Prediction of the annual cycle of temperature changes in a stratified lake or reservoir: Mathematical model and user's manual, *M.I.T. Rept. No. 137*, M.I.T., Cambridge, Mass. (1971).
20. Tuck, E.O. and Vanden-Broeck, J.M., A cusp-like free-surface flow due to a submerged source or sink, *J. Aust. Math Soc. Ser. B* 25 (18984) 443–450.
21. Vanden-Broeck, J.M., Schwartz, L.W., and Tuck, E.O., Divergent low-Froude number series expansion of non-linear free-surface flow problems, *Proc. Roy. Soc. London. Ser. A* 361 (1978) 207–224.
23. Wood, I.R. and Lai, K.K., Selective withdrawal from a two-layered fluid, *J. Hyd. Res.* 10 No. 4 (1972) 475–496.
24. Yih, C.S., *Stratified Flows*, Academic Press (1980).